

The Power of
Monadic Second-Order
Transductions

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Introduction

Transductions are operations of the form

(hyper-)graph \mapsto (hyper-)graph

or graph class \mapsto graph class

defined in terms of **logic**.

Applications

- ◆ decidability results
- ◆ replacement for automata/transducers for arbitrary graphs
- ◆ structural reductions

Hypergraphs

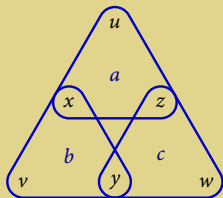
$$G = \langle V, E, \epsilon, P_0, \dots, P_{m-1} \rangle$$

V set of vertices

E set of edges

ϵ incidence relation

P_i colour predicates



$$V = \{u, v, w, x, y, z\}$$

$$E = \{a, b, c\}$$

$$a = \{u, x, z\}$$

$$b = \{v, x, y\}$$

$$c = \{w, y, z\}$$

Monadic second-order logic (MSO)

element variables: x, y, z, \dots

set variables: X, Y, Z, \dots

boolean operations: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$

quantifiers: $\exists x, \forall x, \exists X, \forall X$

Example Reachability

$$\varphi(x, y) := \forall X [x \in X \wedge \forall u \forall v [u \in X \wedge \exists e (u \in e \wedge v \in e) \rightarrow v \in X] \rightarrow y \in X]$$

Transductions

Operations $G \mapsto \tau(G)$ on (hyper-)graphs

$$\tau = \text{int} \circ \text{copy}_k \circ \text{exp}_m$$

- ◆ exp_m expansion by m unary predicates
- ◆ copy_k $G \mapsto G \oplus \dots \oplus G$
- ◆ int MSO-interpretation

$$\langle \chi, \delta_1(x), \delta_2(x), \varphi(x, y), \psi_1(x), \psi_2(x), \dots \rangle$$

$$G \mapsto \langle \delta_1^G, \delta_2^G, \varphi^G, \psi_1^G, \psi_2^G, \dots \rangle \quad (\text{provided } G \models \chi)$$

Graph classes

$$\tau(\mathcal{C}) := \bigcup_{G \in \mathcal{C}} \tau(G)$$

Examples

$\tau_n : \{\text{paths}\} \rightarrow \{\text{trees of height } n\}$



$\chi :=$ “the result is a tree“

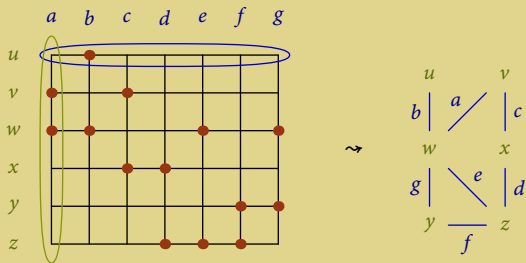
$\delta_1(x) := \text{true}$

$\delta_2(x) := \text{true}$

$\varphi(x, y) :=$ „ x is the right vertex of y .“

\vee „ x is the first vertex to the left of y with the right colour.“

$\tau : \{\text{grids}\} \rightarrow \{\text{graphs}\}$



$\chi := "P_{\bullet}$ forms a column and P_{\bullet} a row."

$\delta_1(x) := P_{\bullet}x$

$\delta_2(x) := P_{\bullet}x$

$\varphi(x, y) := \exists z[P_{\bullet}z \wedge "z$ is in the row of x and in the column of y ."

Theories

Interpretation Lemma

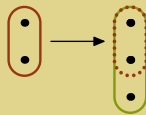
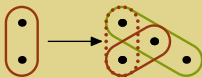
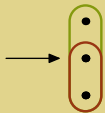
For every sentence $\varphi \in \text{MSO}$ and every transduction τ , there is a sentence $\varphi^\tau \in \text{MSO}$ such that

$$\tau(G) \models \varphi \quad \text{iff} \quad G \models \varphi^\tau \quad \text{for all hypergraphs } G.$$

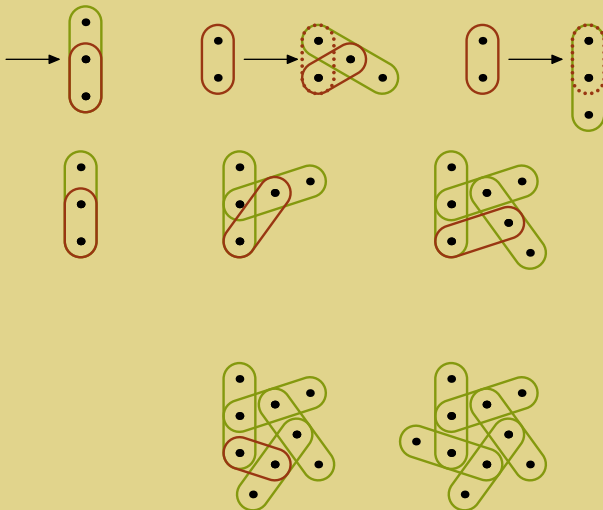
Corollary

Let τ be a transduction. If \mathcal{C} is a class of hypergraphs with **decidable monadic theory**, the theory of $\tau(\mathcal{C})$ is also **decidable**.

Graph grammars



Graph grammars



Derivation trees

Each hypergraph has an **derivation tree**.

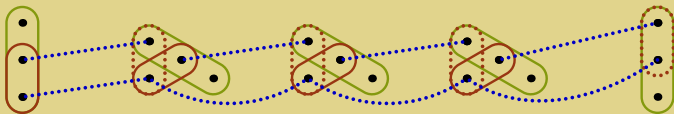
In the example:

$$0 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 2$$

Theorem

For each grammar, there exists a transduction

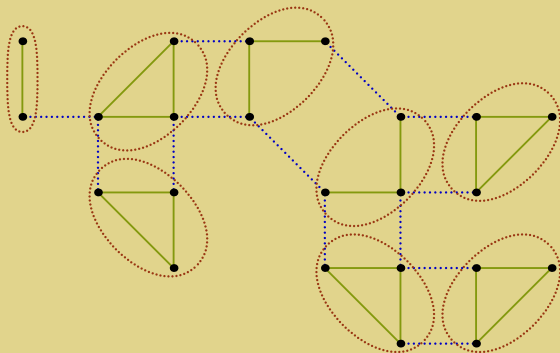
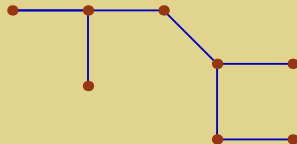
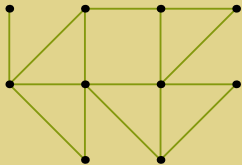
$$\{\text{derivation trees}\} \rightarrow \{\text{hypergraphs}\}$$



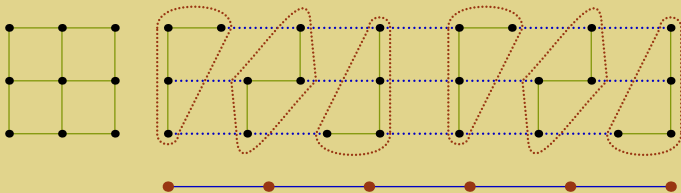
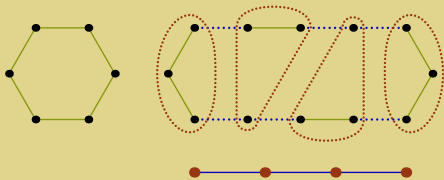
Corollary

Every grammar defines the image of a regular class of finite trees under a transduction.

Tree decompositions



Examples



Tree width

The **width** of a tree decomposition $(U_v)_{v \in T}$ is

$$\text{wd}(U_v)_{v \in T} := \max_{v \in T} |U_v|$$

$\text{twd}(G)$ min. width of a tree decomposition with an **arbitrary tree** as index

$\text{twd}_n(G)$ min. width of a tree decomposition with a **tree of height** $< n$ as index

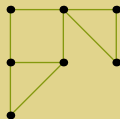
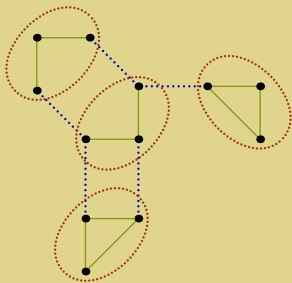
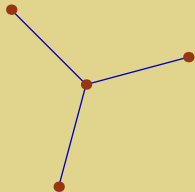
$\text{pwd}(G)$ min. width of a tree decomposition with a **path** as index

Examples

- ◆ **Trees** have tree width 2 and unbounded path width.
- ◆ **Trees of height** n have path width $n + 1$.
- ◆ **Cycles** have tree width and path width 3 .
- ◆ The $n \times n$ **grid** has tree width and path width $n + 1$.

Theorem

For every $n < \omega$, there ex. a **transduction** τ_n , mapping a **tree** T onto the class of all hypergraphs having a **tree decompositions** with **width** $\leq n$ and **index tree** T .



Theorem

For every **transduction** τ , there ex. $n < \omega$, such that every **tree** T is mapped by τ to a hypergraph G having a **tree decomposition** with **width** $\leq n$ and **index tree** T .

Corollary

A class of hypergraphs has **bounded tree width** if, and only if, it is the **image** of a class of **trees** under a **transduction**.

Question What about the converse? Can one obtain the tree decomposition from the hypergraph?

Fact

There is no transduction τ mapping a hypergraph G to the class of **all** tree decompositions of G .

Open Problem

What about computing only some tree decompositions of G ?

Transduction hierarchy

Goal

Classification of classes of **finite hypergraphs** by their **monadic theories**

Order

$\mathcal{C} \leq \mathcal{K}$:iff ex. transduction τ with $\mathcal{C} \subseteq \tau[\mathcal{K}]$

» \mathcal{K} is more complicated than \mathcal{C} .«

$\mathcal{C} \dashv \mathcal{K}$:iff $\mathcal{C} < \mathcal{K}$ and no $\mathcal{C} < \mathcal{D} < \mathcal{K}$

Theorem

$$\emptyset \dashv \mathcal{T}_0 \dashv \mathcal{T}_1 \dashv \cdots \dashv \mathcal{T}_n \dashv \cdots < \mathcal{P} \dashv \mathcal{T}_\omega \dashv \mathcal{G}$$

\mathcal{T}_n trees of height $< n$

\mathcal{T}_ω all trees

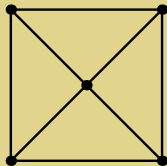
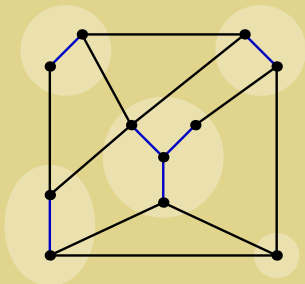
\mathcal{P} all paths

\mathcal{G} all grids

Graph minors

Definition

A **minor** is obtained by: **deletion** of vertices and edges, **contraction** of edges.



Lemma

Ex. transduction $\tau : G \mapsto \text{Min}(G)$.

Excluded Grid Theorem

Theorem (Robertson, Seymour)

(a) For every tree T , ex. $k < \omega$ such that

$$T \notin \text{Min}(G) \Rightarrow \text{pwd}(G) < k$$

(b) For every grid E , ex. $k < \omega$ such that

$$E \notin \text{Min}(G) \Rightarrow \text{twd}(G) < k$$

Theorem

For every path P , ex. $n, k < \omega$ such that

$$P \notin \text{Min}(G) \Rightarrow \text{twd}_n(G) < k$$

Consequences

Theorem

$$\mathcal{C} \leq \mathcal{P} \quad \text{iff} \quad \text{pwd}(\mathcal{C}) < \infty$$

$$\mathcal{C} \leq \mathcal{T}_\omega \quad \text{iff} \quad \text{twd}(\mathcal{C}) < \infty$$

$$\mathcal{C} \leq \mathcal{T}_n \quad \text{iff} \quad \text{twd}_n(\mathcal{C}) < \infty$$

Corollary

$$\mathcal{P} \dashv \mathcal{T}_\omega \dashv \mathcal{G}$$

$$\mathcal{P} \not\leq \mathcal{C} \Rightarrow \mathcal{C} \leq \mathcal{T}_n \text{ for some } n$$

Proof

$$\mathcal{G} \not\leq \mathcal{C} \Rightarrow \mathcal{G} \notin \text{Min}(\mathcal{C}) \Rightarrow \text{twd}(\mathcal{C}) < \infty \Rightarrow \mathcal{C} \leq \mathcal{T}_\omega$$

The lower part of the hierarchy

Lemma

For every $n < \omega$, ex. transduction τ_n mapping a hypergraph G onto the class of all trees of height $\leq n$ that are the index tree of a **strict** tree decomposition of G .

Lemma

For every tree decomposition, there ex. a strict tree decomposition of the same width and height.

Corollary

Every $\mathcal{C} \leq \mathcal{T}_n$ is equivalent to a subclass of \mathcal{T}_n .

Theorem

$$\mathcal{T}_{n+1} \not\subseteq \text{Min}(\mathcal{C}) \Rightarrow \text{twd}_n(\mathcal{C}) < \infty \quad \text{for } \mathcal{C} \subseteq \mathcal{T}_\omega.$$

Corollary

$$\mathcal{C} < \mathcal{T}_{n+1} \Rightarrow \mathcal{C} \leq \mathcal{T}_n$$

Theorem

For every transduction τ , ex. $k < \omega$ such that

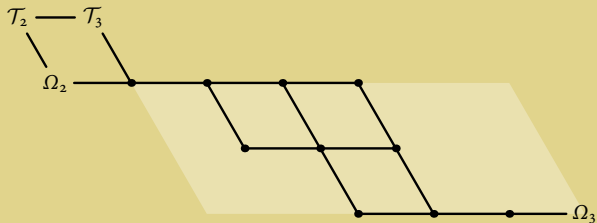
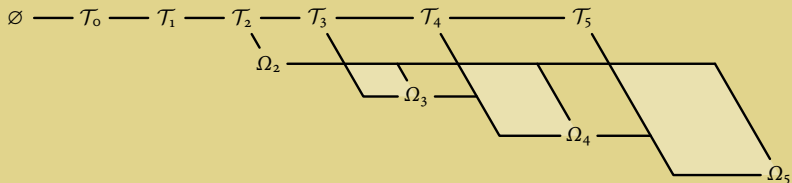
$$\Delta(\tau(T)) \geq \frac{\Delta(T)}{k} \quad \text{for trees } T, \tau(T)$$

Corollary

$$\mathcal{T}_{n+1} \not\leq \mathcal{T}_n$$

(counting argument)

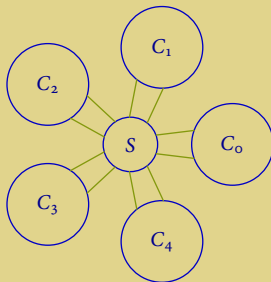
Hierarchy for countable hypergraphs



Orderable classes

Question: Is there a transduction $G \mapsto \langle G, < \rangle$, for all $G \in \mathcal{C}$?

An obvious obstruction



$\text{Sep}(G, k)$ is the maximal number of connected components of $G - S$, for some $S \subseteq V$ with $|S| \leq k$.

\mathcal{C} has $\text{SEP}(f)$ if $\text{Sep}(G, k) \leq f(k)$, for all k .

Lemma

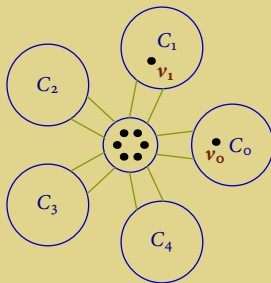
If \mathcal{C} is orderable, it has $\text{SEP}(f)$ for some $f : \omega \rightarrow \omega$.

Proof sketch

$S = \{s_1, \dots, s_k\}$, C_0, C_1 connected components of $G - S$, $v_i \in C_i$

$$C_0 \cup S \models \varphi(v_0; s_1, \dots, s_k) \quad \text{iff} \quad C_1 \cup S \models \varphi(v_1; s_1, \dots, s_k)$$

implies $G \models \psi(v_0, v_1) \quad \text{iff} \quad G \models \psi(v_1, v_0)$.



Lemma

There are formulae $\varphi_{k,d}(x, y)$ ordering every graph G with $K_{k,k} \not\leq G$ and $\text{Sep}(G, k) \leq d$.

Theorem

Let \mathcal{C} be a class omitting some minor. Then \mathcal{C} is orderable if, and only if, it has $\text{SEP}(f)$ for some f .

Encoding edges by vertices

Representations of a graph G :

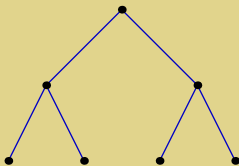
- two-sorted $G_{\text{in}} = \langle V, E, \text{in} \rangle$ with $\text{in} \subseteq V \times E$
- one-sorted $G_{\text{ad}} = \langle V, \text{adj} \rangle$ with $\text{adj} \subseteq V \times V$

Question Is $G_{\text{ad}} \mapsto G_{\text{in}}$ a transduction?

Obviously, this is only possible if G has few edges.

Example

For a tree T encode an edge $u \rightarrow v$ by the vertex v .



Definition A graph G is k -sparse if

$$|E \upharpoonright X| \leq k|X| \quad \text{for all } X \subseteq V.$$

Theorem

For every k , there exists a transduction τ_k with

$$\tau_k(G_{\text{ad}}) = G_{\text{in}} \quad \text{for every } k\text{-sparse } G.$$