

# Composition for Orders with an Extra Binary Relation

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**Bruno's Workshop, Bordeaux, June 2012**

# Two Traditions in Effective Logic

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- Orderings with unary predicates  
MSO-logic  
Automata, Composition method
- Graphs  
Courcelle theory (for MSO)  
Hanf and Gaifman theorems (for FO)

Here: Study of **FO-theory of orderings expanded by graphs**

Structures:  $(\mathbb{N}, <, R)$  with binary  $R$

Restriction:  $R$  is finite valency

$R \subseteq A \times A$  is of **finite valency** if for each  $a \in A$  there are only finitely many  $b \in A$  with  $R(a, b)$  or  $R(b, a)$ .

Injective functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  provide examples.

1. **MSO-Th( $\mathbb{N}, <, P$ ) for unary  $P$**
2. **Homogeneity of colorings is first-order definable**
3. **Orderings ( $\mathbb{N}, <, R$ ) with binary  $R$**
4. **Conclusion**

# Structures $(\mathbb{N}, <, P)$ with unary $P$

Identify  $P \subseteq \mathbb{N}$  with 0-1-word  $\alpha(P)$

Consequence of Büchi's analysis of  $\text{MSO-Th}(\mathbb{N}, <)$ :

**MSO-Th $(\mathbb{N}, <, P)$  is decidable iff the following decision problem is decidable:**

**Given a Büchi automaton  $\mathcal{A}$ , decide whether  $\mathcal{A}$  accepts  $\alpha(P)$ .**

**So one only needs to decide whether the word  $\alpha(P)$  can be cut into pieces  $u_0, u_1, \dots$  such that**

**$\mathcal{A} : q_0 \xrightarrow{u_0} q$  and  $\mathcal{A} : q \xrightarrow{u_i} q$  for  $i = 1, 2, \dots$ , with  $q$  final.**

**The composition method allows this reduction to periodicity directly, without reference to automata.**

# $m$ -Types (for FO and MSO)

Given quantifier-depth  $m$  define for two words  $u, v$   
(finite or infinite!):

$$u \equiv_m v : \iff$$

$u$  and  $v$  satisfy the same sentences of quantifier-depth  $m$

Facts:

- $\equiv_m$  is an equivalence relation of finite index;  
call the equivalence classes  **$m$ -types**.
- An  $m$ -type  $\tau$  is definable by a sentence  $\varphi_\tau$  of  
quantifier-depth  $m$ .
- Each sentence  $\psi$  of quantifier-depth  $m$  is equivalent to a  
disjunction of sentences  $\varphi_\tau$ .

# Composition

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1. From the  $m$ -types of  $u$  and  $v$  one can compute the  $m$ -type of  $uv$ .
2. From the  $m$ -type of  $u$  one can compute the  $m$ -type of  $uuu\dots$

**Consequence:**

**Given  $\alpha = uvvv\dots$ , the  $m$ -type  $\varrho$  of  $\alpha$  is determined by the  $m$ -types  $\sigma$  of  $u$  and  $\tau$  of  $v$ ;**

**we write  $\varrho = \sigma + \sum_{\omega} \tau$**

**Ramsey's Theorem guarantees such a decomposition for arbitrary  $\alpha$**

# Finite Colorings

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Given a finite set  $\text{Col} = \{c_1, \dots, c_r\}$  of colors.

A **coloring** over  $\mathbb{N}$  with  $\text{Col}$  is a map

$$C : \{(m, n) \mid m < n\} \rightarrow \text{Col}$$

$C$  is **additive** if from  $C(\ell, m) = C(\ell', m')$  and  $C(m, n) = C(m', n')$  we can infer  $C(\ell, n) = C(\ell', n')$ .

For colors  $c, d$  we may write  $c + d$ .

**Example:** For quantifier-depth  $m$  and  $\omega$ -word  $\alpha$  define

$$C_\alpha^m(i, j) = m\text{-type of } \alpha[i, j - 1]$$

(either for FO or MSO)

# Ramsey's Theorem

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For any finite additive coloring  $C$  there is a "homogeneous set"  $H = \{h_0 < h_1 < h_2 < \dots\}$  such that all colors  $C(h_i, h_j)$  (where  $i < j$ ) coincide.

**Consequence:** Then there are two colors  $c, d$  such that

$$C(0, h_0) = c \text{ and } C(h_i, h_{i+1}) = d$$

Call a color pair  $(c, d)$  **good for  $C$**  if there is  $H = \{h_0 < h_1 < \dots\}$  such that  $C(0, h_0) = c$  and  $C(h_i, h_j) = d$  for  $i < j$ ,

in particular,  $C(h_i, h_{i+1}) = d$ , and  $d = d + d$ .



# Back to MSOTh( $\mathbb{N}, <, P$ )

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**MSOTh( $\mathbb{N}, <, P$ ) is decidable**

**iff for each  $m$  we can compute the  $m$ -type  $\rho$  of  $\alpha(P)$**

**iff for each  $m$  and the associated coloring  $C_{\alpha(P)}^m$  we can compute those pairs  $(\sigma, \tau)$  of  $m$ -types which are good for  $C_{\alpha(P)}^m$ .**

**In other words, for any  $P$ :**

**A sentence  $\psi$  of quantifier-depth  $m$  is effectively equivalent over  $(\mathbb{N}, <, P)$  to a disjunction of statements " $(\sigma, \tau)$  is good for  $C_{\alpha(P)}^m$ "**

**[Compare with the automata theoretic periodicity condition.]**

# Defining to be Good

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Let  $\bar{C}$  be the tuple of binary predicates " $C(i, j) = c$ ".

Consider the associated structure  $(\mathbb{N}, <, \bar{C})$ .

Remark:

There is an MSO-sentence  $\varphi_{c,d}$  saying in  $(\mathbb{N}, <, \bar{C})$  that  $(c, d)$  is good for  $C$ :

$\exists X$  ( $X$  is infinite

$\wedge C(0, x) = c$  for the smallest element  $x$  of  $X$

$\wedge C(x, y) = d$  for any  $x, y \in X$  with  $x < y$ )

We show that an FO-sentence suffices.

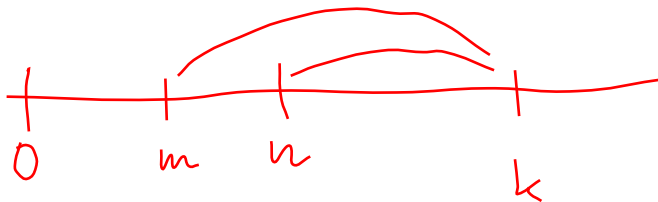
This will also give a proof of Ramsey's Theorem.

# McNaughton's Merge-Relation

Given  $\alpha$  and an additive coloring  $C$ .

$m, n$  **merge at  $k$**  (short  $m \sim_C n(k)$ ) if  $C(m, k) = C(n, k)$

If  $m, n$  merge at  $k$  then also at each  $k' > k$ .



# Lemma 1

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$(c, d)$  is good for  $C$  iff

$$(*) \quad \exists n [C(0, n) = c \wedge \forall m \exists k > m (C(n, k) = d \wedge n \sim_C k)]$$

Show  $\Leftarrow$ :

Take  $n_0$  as the smallest  $n$  according to  $(*)$ .

Assume  $n_0, \dots, n_i$  are defined, with  $n_0 \sim_C n_j$  for  $j = 1, \dots, i$ .

Let  $n_0, \dots, n_i$  merge at  $m$ .

Define  $n_{i+1}$  as the smallest number  $k > m$  guaranteed by  $(*)$ , namely with  $C(n_0, n_{i+1}) = d$  and  $n_0 \sim n_j$  for all  $j = 1, \dots, i + 1$ .

# Consequence: Ramsey's Theorem

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Let  $M$  be an infinite  $\sim_C$ -equivalence class

Let  $n_0$  be its smallest element and set  $c = C(0, n_0)$ .

For some  $d$  infinitely many  $n$  in  $M$  exist with  $C(n_0, n) = d$

Then  $(*)$  is satisfied.

Hence  $(c, d)$  is good for  $C$ .

# Reducing Quantifier Alternation

(\*)  $\exists n[C(0, n) = c \wedge \forall m \exists k > m(C(n, k) = d \wedge n \sim_C k)]$

is a  $\Sigma_3$ -condition (w.r.t. unbounded quantifiers).

Show that it can be written as a Boolean combination of  $\Sigma_2$ -conditions.

Define a set  $M_{\ell, c}(x)$ :

Consider the  $\ell$ -tuples of distinct numbers  $n_1, \dots, n_\ell \leq x$  such that  $C(0, n_i) = c$  and any two of the  $n_j$  do not merge at  $x$ .

- If such an  $\ell$  tuple exists  
let  $M_{\ell, c}(x)$  contain the elements of the smallest such tuple  
(in lexicographical ordering)
- otherwise let  $M_{\ell, c}(x) = \{x\}$ .

# Lemma 2

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## Define

- $g_{\ell,c}(x) = \max M_{\ell,c}(x)$
- $f_{\ell,c,d}(x) =$  the greatest  $y < x$  such that for some  $z \in M_{\ell,c}$   
 $C(z, y) = d$  and  $C(y, x) = d$  and  $C(z, x) = d$   
(take value 0 if such  $y$  does not exist)

## Then

$(c, d)$  is good for  $C$  iff

$\forall_{\ell=1}^r (g_{\ell,c}$  is bounded and  $f_{\ell,c,d}$  unbounded).

# The $\forall\exists\wedge\exists\forall$ -Lemma

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Let  $C$  be an additive finite coloring and  $\bar{C}$  be the tuple of relations  $C(i, j) = c$ .

There are bounded formulas  $\varphi_{c,\ell}(y)$  and  $\psi_{\ell,c,d}(y)$  such that  $(c, d)$  is good for  $C$  iff

$$(\mathbb{N}, <, \bar{C}) \models \bigvee_{\ell=1}^{|\bar{C}|} (\exists x \forall y > x \varphi_{c,\ell}(y) \wedge \forall x \exists y > x \psi_{\ell,c,d}(y))$$

**Application: McNaughton's Theorem**

**Any Büchi automaton can be converted into a deterministic Muller automaton.**

Use  $\sim_{\mathcal{A}}$ -classes as colors:  $u \sim_{\mathcal{A}} v$  iff for any states  $p, q$

$$\mathcal{A} : p \xrightarrow{u} q \text{ [passing } F] \Leftrightarrow \mathcal{A} : p \xrightarrow{v} q \text{ [passing } F]$$



# Other Applications

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1. For any  $P \subseteq \mathbb{N}$ : **MSO-Th** $(\mathbb{N}, <, P)$  is decidable iff **WMSO-Th** $(\mathbb{N}, <, P)$  is.
2. Any FO-definable  $\omega$ -language can be recognized by a counter-free Muller automaton.

# Binary relations and composition

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Consider structures  $(\mathbb{N}, <, R)$  with binary  $R$

The  $m$ -types of two segments  $([\ell, m], <, R|_{[\ell, m]})$  and  $([m, n], <, R|_{[m, n]})$  are not sufficient to determine the  $m$ -type of  $([\ell, n], <, R|_{[\ell, n]})$

But we can do composition if enough interface information is provided.

# Finite Valency

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Let  $R \subseteq \mathbb{N} \times \mathbb{N}$  be of finite valency: For any  $a$  there are at most finitely many  $b$  with  $R(a, b)$  or  $R(b, a)$ .

Call  $[a, b]$  an  **$R$ -segment** if  $R(a, b)$  or  $R(b, a)$ .

An  $R$ -segment is **maximal** if it is not properly contained in another  $R$ -segment.

Remark: If  $R$  is of finite valency then each  $R$ -segment is contained in a maximal one.

# $m$ -Admissible Segments

Define for each  $b$  a sequence  $b(0) > b(1) > \dots$  as follows:

$$b(0) = b$$

$$b(i+1) =$$

- biggest  $c$  which is below all maximal  $R$ -segments  $[k, \ell]$  intersecting  $[b(i), \infty)$ , if such  $c$  exists,
- 0 otherwise

The segment  $[a, b]$  is  **$m$ -admissible** if  $b(2^m) > a$

Write  $b^*$  for  $b(2^m)$  if  $m$  is clear.

Denote by  $\tilde{b}$  the sequence  $(b(0), \dots, b(2^m))$ .

For any  $k$  there exist admissible segments  $[a, b]$  above  $k$ .

# $T$ - and $D$ -Types

Let  $[a, b]$  be  $m$ -admissible,  $a_0, \dots, a_{r-1} \in [a, b]$ .

let

- $T_R^m[a, b](a_0, \dots, a_{r-1})$  be the FO- $m$  type of the restriction of  $\mathbb{N}$  to  $[a, b]$
- $D_R^m[a, b](a_0, \dots, a_{r-1}) := T_R^m[a^*, b](\tilde{a}, \tilde{b}, a_0, \dots, a_{r-1})$

$D_R^m$  defines an almost total coloring:

For each  $a$  there are only finitely many  $b \geq a$  such that  $[a, b]$  is not  $m$ -admissible.

# Composition Lemma

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1. Given  $m$ -admissible segments  $[a, b]$  and  $[b, c]$ ,  
 $D_R^m[a, b]$  and  $D_R^m[b, c]$  determine effectively the type  
 $D_R^m[a, c]$
2. Given a sequence  $a_0, a_1, \dots$  such that  $[a_i, a_{i+1}]$  is  
 $m$ -admissible and  $D_R^m[a_i, a_{i+1}] = \tau$  for some  $m$ -type  $\tau$ ,  
 $D_R^m[a_0, \infty)$  is determined effectively by  $\tau$ .

If  $D_R^m[0, a_0] = \sigma$  we may write

$$D_R^m[0, \infty) = \sigma + \tau + \tau + \dots$$

# Nondefinability of $+$ and $\cdot$

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## Theorem:

In a structure  $(\mathbb{N}, <, R)$  with  $R$  of finite valency, neither addition nor multiplication is FO-definable.

## Lemma:

Let  $f : \mathbb{N}^2 \rightarrow \mathbb{N}$  be FO-definable in  $(\mathbb{N}, <, R)$  where  $R$  is of finite valency.

Then one of the following two sets is finite:

$$X_f := \{x \in \mathbb{N} \mid \lambda y f(x, y) \text{ is injective}\}$$

$$Y_f := \{y \in \mathbb{N} \mid \lambda x f(x, y) \text{ is injective}\}.$$

**Note:**  $X_+, Y_+, X_\cdot, Y_\cdot$  are all infinite.

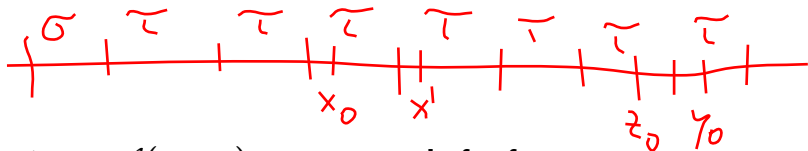
# Proof of Lemma

Assume  $f$  is FO-definable by  $\varphi(x, y, z)$  of quantifier-depth  $m$

Consider the coloring  $C_R^{m+1}$  and a homogeneous set  $H$

Pick  $x_0 \in X_f, y_0 \in Y_f$  in distant  $H$ -segments:

$$x_0 \in [h_i, h_{i+1}], y_0 \in [h_j, h_{j+1}]$$



Let  $z_0 = f(x_0, y_0)$ , assume  $z_0$  is far from  $x_0$

Pick  $x' \in [h_{i+1}, h_{i+2}]$  of same  $m$ -type as  $x_0$  in  $[h_i, h_{i+1}]$

Then  $f(x_0, y_0) = f(x', y_0) = z_0$

$\lambda y f(x_0, y)$  not injective, so  $y_0 \notin Y_f$



# A Normal Form

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Over  $(\mathbb{N}, < R)$  each sentence  $\varphi$  is equivalent to a sentence

$$\bigvee_{i=1}^n (\forall x \exists y \forall z \varphi_i(x, y, z) \wedge \exists x \forall y \exists z \psi_i(x, y, z))$$

where the  $\varphi_i, \psi_i$  are bounded in  $z$ .

Use the  $\forall \exists \wedge \exists \forall$ -Lemma for additive colorings.

As color formulas  $C(x, y) = c$  we take  $D_R^m[x, y] = \tau$

We can define  $D_R^m[x, y] = \tau$  relative to a bound  $z$  by a formula  $D_R^m[x, y](z) = \tau$  bounded in  $z$ .

Then:

$$D_R^m[x, y] = \tau$$

$$\Leftrightarrow \exists^\omega z D_R^m[x, y](z) = \tau$$

$$\Leftrightarrow \forall^\omega z D_R^m[x, y](z) = \tau$$

# Degree of FO-Th( $\mathbb{N}, <, R$ )

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For  $R \subseteq \mathbb{N}^2$  of finite valency,  $\text{FO-Th}(\mathbb{N}, <, R) \leq_T R'''$ ,  
and  $R'''$  cannot in general be replaced by  $R''$

**Proof of lower bound:**

$V_3 := \{m \mid \forall k \exists \ell > k (\ell \in W_m \wedge W_\ell = \emptyset)\}$  (is  $\Pi_3$ -complete)

**So:**

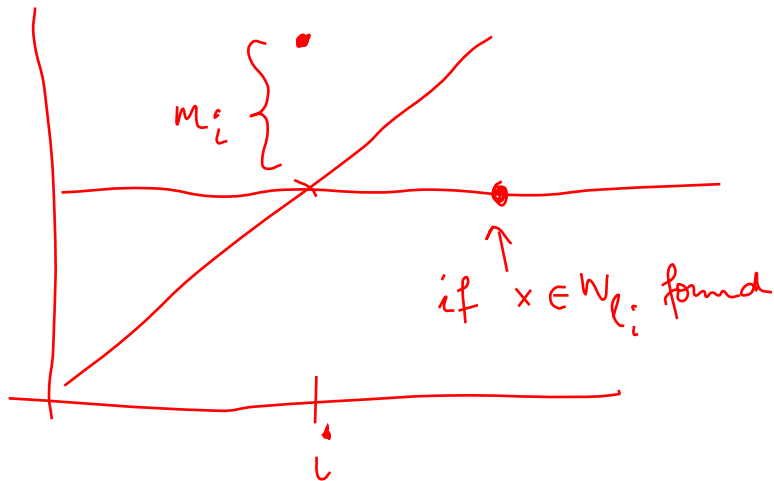
$m \in V_3$

$\Leftrightarrow W_m$  contains infinitely many indices of the empty r.e. set

**Find recursive  $R$  of finite valency s.t. a  $V_3 \leq_T \text{FO-Th}(\mathbb{N}, <, R)$**

**Let  $(\ell_i, m_i)$  the  $i$ -th pair in enumeration of all  $(\ell, m)$  with  $\ell \in W_m$**

# Explaining the Reduction



# Reduction

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Recall:  $(\ell_i, m_i)$  is the  $i$ -th pair in enumeration of all  $(\ell, m)$  with  $\ell \in W_m$

- At stage  $i$  include  $(i, i + m_i + 1)$  in  $R$ ,
- Using  $\ell_i$  include  $(i, i)$  if up to stage  $i$  an element in  $W_{\ell_i}$  was found,
- At later stage  $j$  include  $(j, i)$  if then for the first time an element in  $W_{\ell_i}$  is found.

$R$  is recursive and of finite valency.

$$\exists^\omega \ell (\ell \in W_m \wedge W_\ell = \emptyset)$$

$$\text{iff } (\mathbb{N}, <, R) \models \exists^\omega x (R(x, x + m + 1) \wedge \forall z \neg R(z, x))$$

# Special Case

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$R$  is **effectively of finite valency**

if the finite sets  $N_a = \{b \mid R(a, b)\}$  and  $N_b = \{a \mid R(a, b)\}$  are computable from  $a, b$ , respectively.

In this case the bound  $R'''$  can be replaced by  $R''$ .

# Strongly Monotone Functions

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For a monotone function  $f : \mathbb{N} \rightarrow \mathbb{N}$  define  $\Delta_f$  by  $\Delta_f(n) = f(n+1) - f(n)$ .

Call  $f$  **strongly monotone** if  $f$  and  $\Delta_f$  are monotone (in the  $\leq$ -sense).

## Theorem

If  $f$  is strongly monotone and recursive, the first-order theory of  $(\mathbb{N}, <, f)$  is decidable.

In contrast: Let  $f_{\mathbb{P}}(i) = i$ -th prime.

Is  $\text{FO-Th}(\mathbb{N}, <, f_{\mathbb{P}})$  decidable?

Is  $\text{FO-Th}(\mathbb{N}, <, \mathbb{P})$  decidable?

# Conclusion

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## 1. Variations:

Signatures  $\langle, R_1, \dots, R_n$  rather than  $\langle, R$ .

Finitely many exceptions of finite valency do not matter.

Adaptation to relations of higher arity is possible.

Injective function  $f$  allows definability of  $+$  and  $\cdot$ .

In  $\text{FO-Th}(\omega^2, \langle, R)$  with  $R$  of finite valency  $+$  and  $\cdot$  are definable.

## 2. Questions:

Are there natural examples of recursive  $R$  with  $\text{FO-Th}(\mathbb{N}, \langle, R)$ ?

What about  $\text{FO-Th}(\mathbb{N}, \langle, f_{\mathbb{P}})$ ,  $\text{FO-Th}(\mathbb{N}, \langle, \perp)$ ,  $\text{FO-Th}(\mathbb{N}, +, \perp)$  ?

Are there applications in model-checking?

# Summary

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- **Composition of FO-types is possible for segments of  $\mathbb{N}$  if the order  $<$  is expanded by a binary relation  $R$  of finite valency.**
- **In this case, the structure  $(\mathbb{N}, <, R)$  is ultimately periodic when analyzing it w.r.t. fixed quantifier-depth.**
- **This excludes definability of  $+$  and  $\cdot$ .**
- **Any FO-sentence is expressible as a Boolean combination of  $\Sigma_3$ -sentences (with bounded kernel).**